Volumes by Cylindrical Shells (Silindirik Kabuk ile Hacim Bulma)

In this section, we will learn:

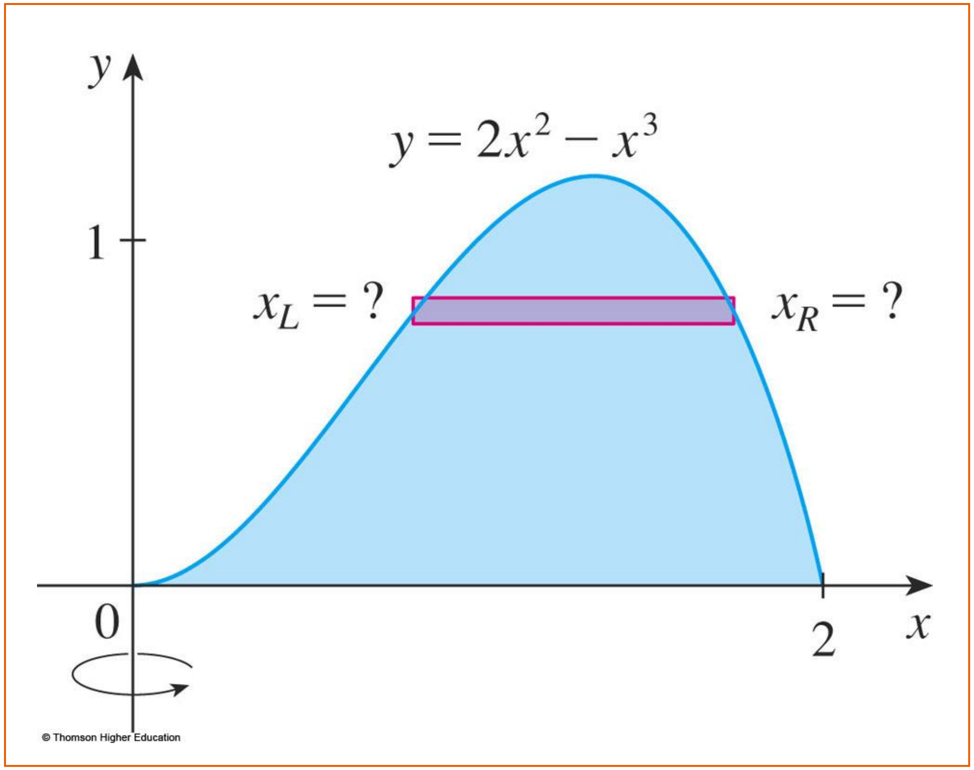
How to apply the method of cylindrical shells to find out the volume of a solid.

VOLUMES BY CYLINDRICAL SHELLS

Bazı hacim problemlerini önceki yöntemlerle (Disk, Pul) ele almak zordur.

VOLUMES BY CYLINDRICAL SHELLS

Let's consider the problem of finding the volume of the solid obtained by rotating about the *y*-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

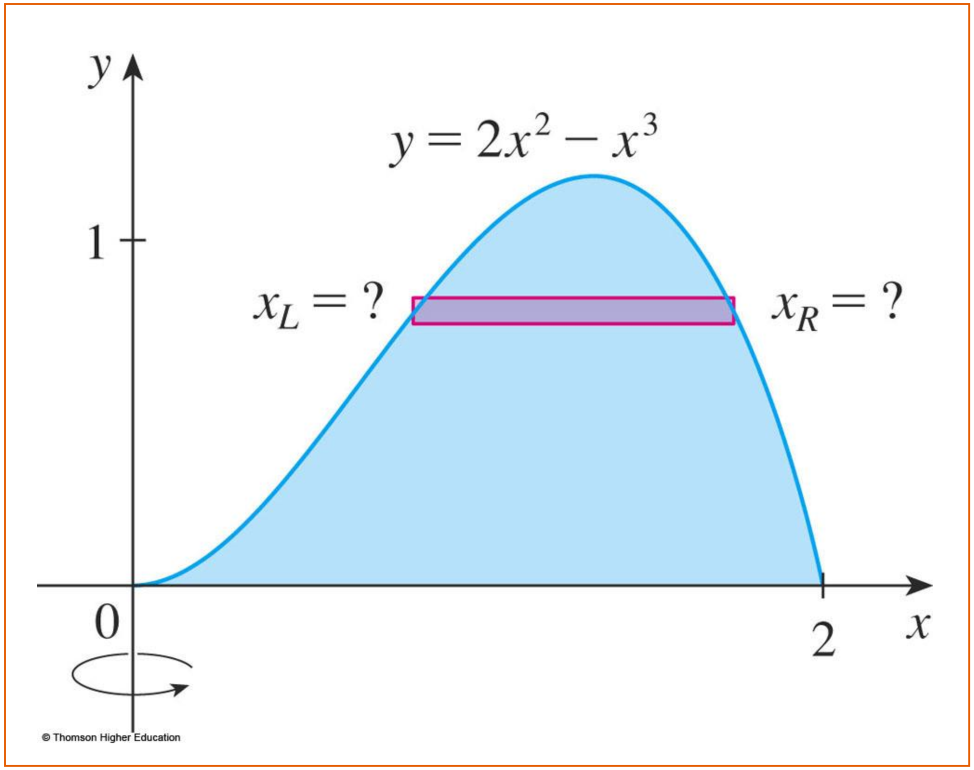


VOLUMES BY CYLINDRICAL SHELLS

If we slice perpendicular to the *y*-axis, we get a washer.

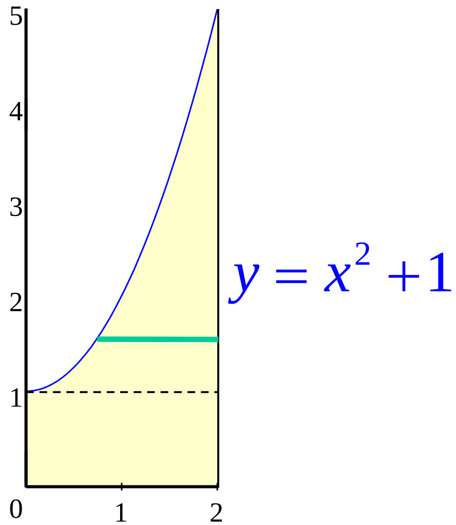
- However, to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y.

- That's not easy.



Fortunately, there is a method—the method of cylindrical shells—that is easier to use in such a case.

VOLUMES BY CYLINDRICAL SHELLS



$$y - 1 = x^2 \qquad x = \sqrt{y-1}$$

$$\pi \int_1^5 \left(2^2 - \left(\sqrt{y-1} \right)^2 \right) dy + \pi \cdot 2^2 \cdot 1$$

outer radius

inner radius

thickness of slice

cylinder

$$\pi \int_1^5 \left(4 - \left(y - 1 \right) \right) dy + 4\pi$$

$$\pi \int_1^5 \left(5 - y \right) \, dy + 4\pi$$

$$\pi \left[5y - \frac{1}{2} y^2 \right]_1^5 + 4\pi$$

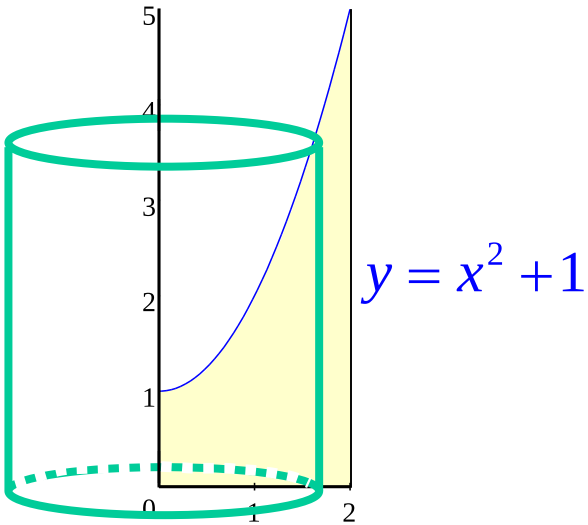
$$\pi \left[\left(25 - \frac{25}{2} \right) - \left(5 - \frac{1}{2} \right) \right] + 4\pi$$

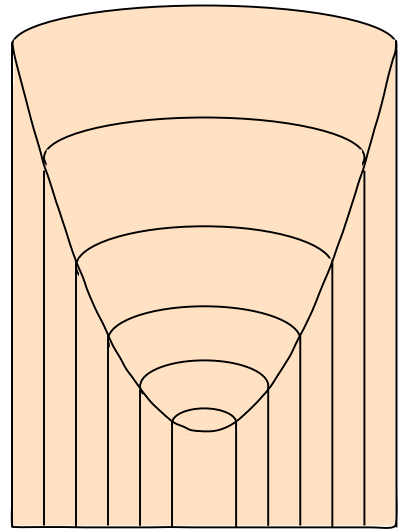
$$\pi \left[\frac{25}{2} - \frac{9}{2} \right] + 4\pi$$

$$\pi \cdot \frac{16}{2} + 4\pi$$

$$8\pi + 4\pi \quad = 12\pi$$

→



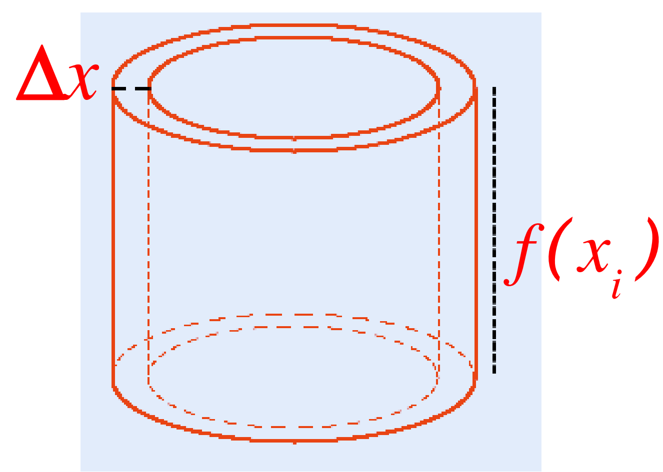


cross section

Here is another way we could approach this problem:

Dikey bir dilim alır ve y ekseni etrafında döndürürsek bir silindir elde ederiz.

Tüm silindirleri birlikte eklersek, orijinal nesneyi yeniden yapılandırabiliriz.



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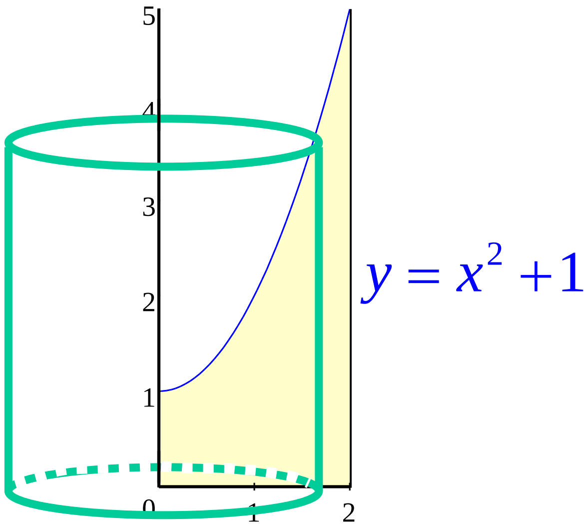
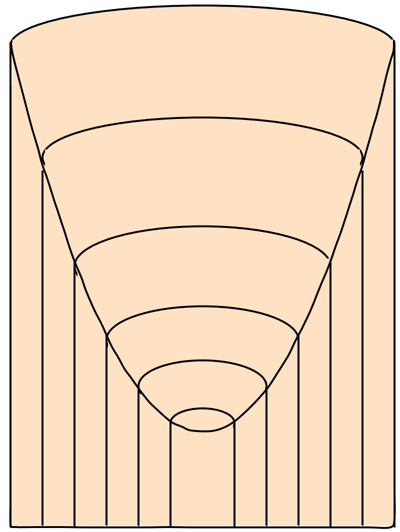




cross section

The volume of a thin, hollow cylinder is given by:

$$\text{Lateral surface area of cylinder} \cdot \text{thickness}$$

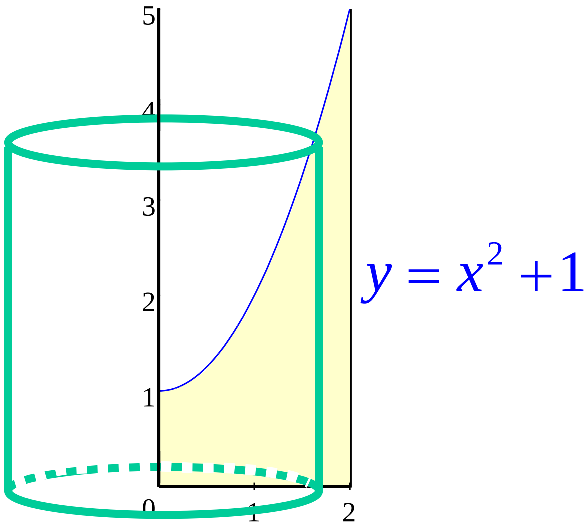
$$= \text{circumference} \cdot \text{height} \cdot \text{thickness}$$

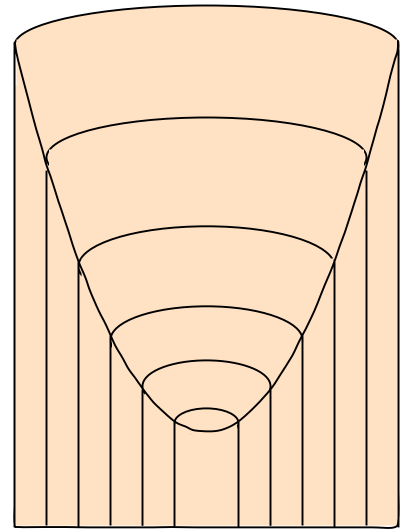
$$= 2\pi r \cdot h \cdot \text{thickness}$$

$$= 2\pi x\left(x^2 + 1\right)\ dx$$

r (circumference: $2\pi x$), h: $\left(x^2+1\right)$, thickness: dx

→





cross section

This is called the <u>shell method</u> because we use cylindrical shells.

If we add all the cylinders from the smallest to the largest:

$$\int_0^2 2\pi x\left(x^2+1\right)\,dx$$

$$=2\pi r\cdot h\cdot \text{thickness}$$

$$=2\pi x\left(x^2+1\right)\ dx$$

r (circumference), h, thickness

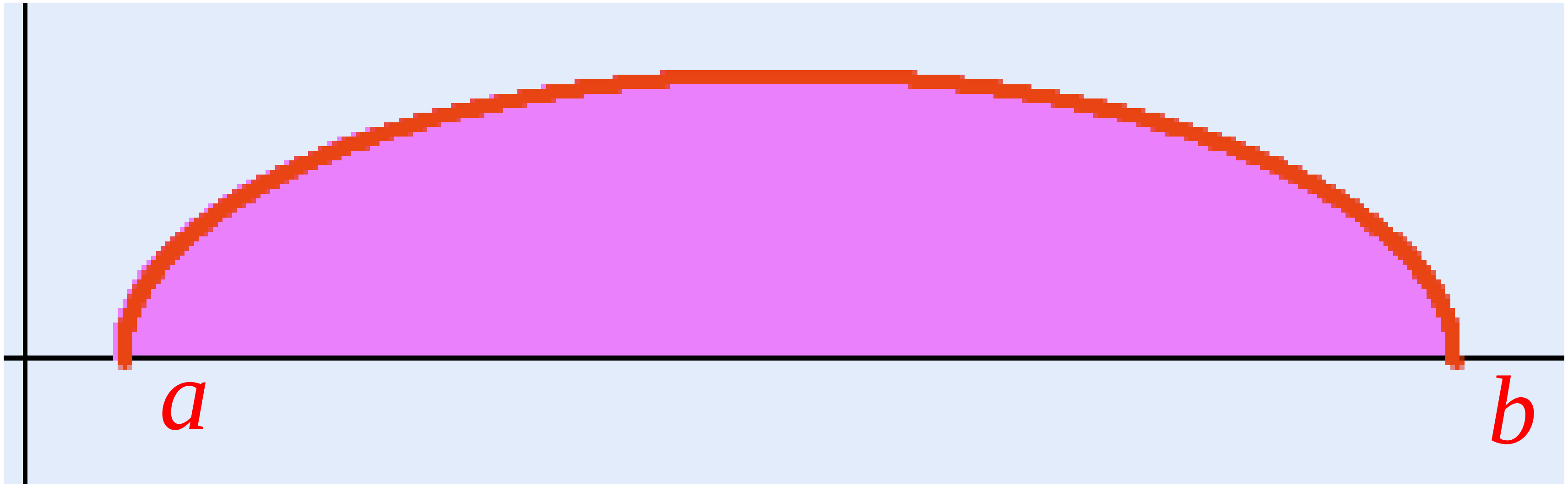
$$2\pi\int_0^2 x^3 + x\ dx$$

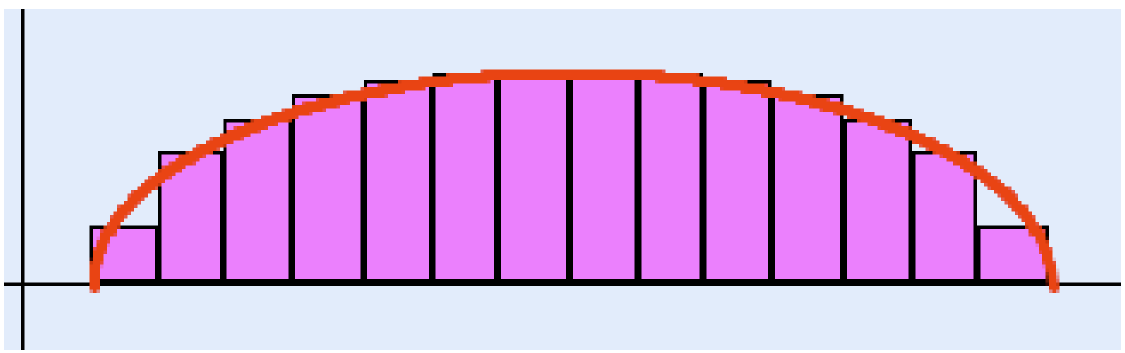
$$2\pi\left[\frac{1}{4}x^4+\frac{1}{2}x^2\right]_0^2$$

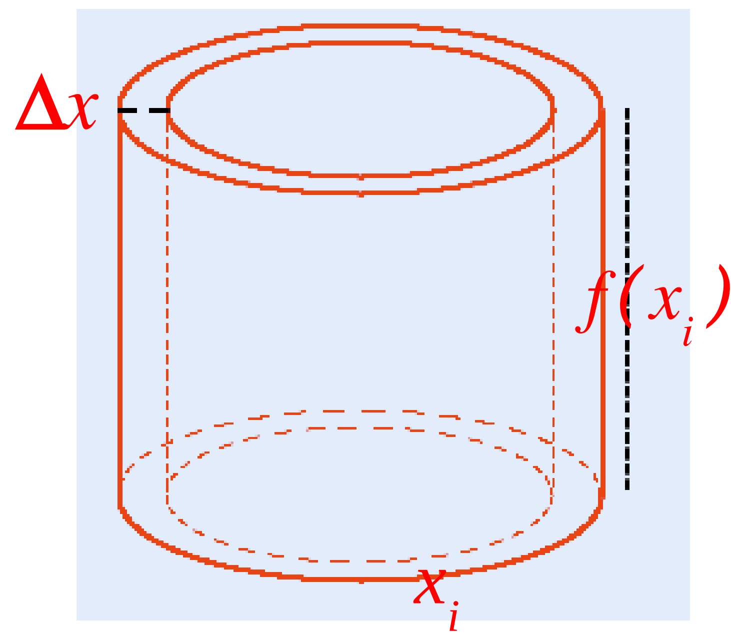
$$2\pi\left[4+2\right]$$

$$12\pi$$

→


a
b



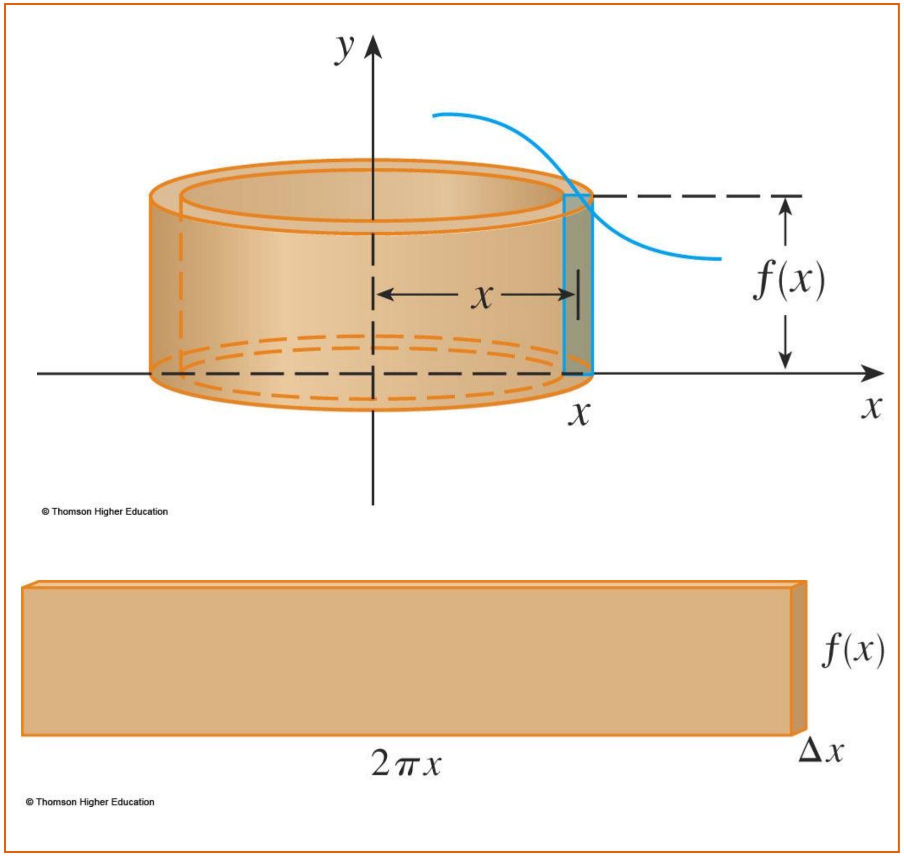

Δx
f(x_i)
x_i

CYLINDRICAL SHELLS METHOD

Here's the best way to remember the formula.

- Think of a typical shell, cut and flattened, with radius *x*, circumference $2\pi x$, height $f(x)$, and thickness Δx or dx:

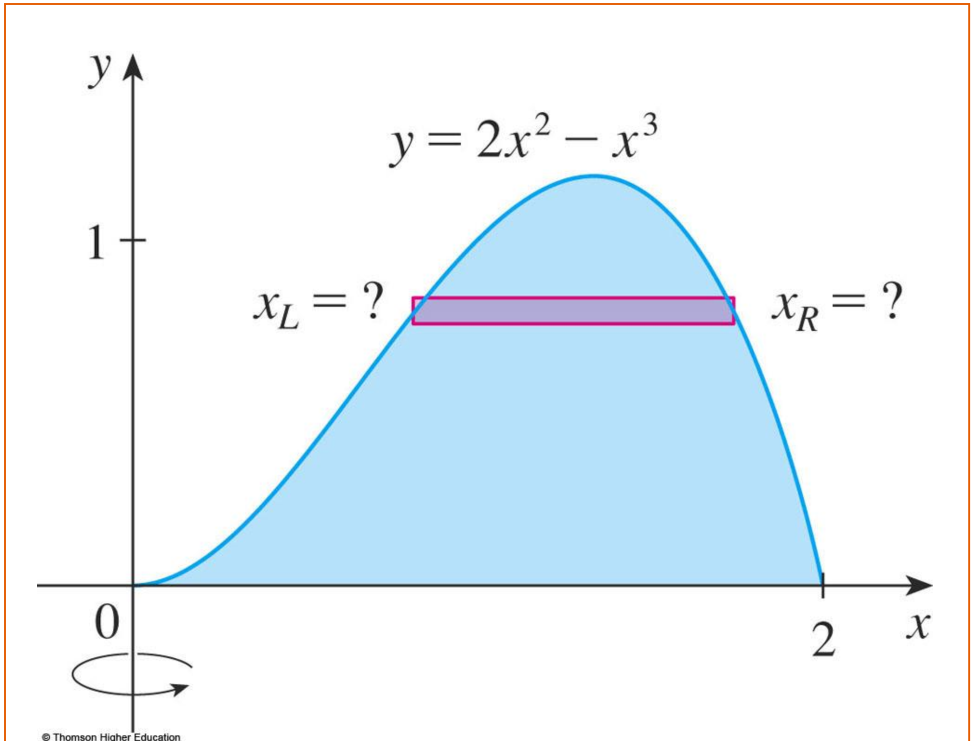
$$\int_a^b \underbrace{(2\pi x)}_{\text{circumference}} \underbrace{[f(x)]}_{\text{height}} \underbrace{dx}_{\text{thickness}}$$



CYLINDRICAL SHELLS METHOD

Example 1

Let's consider the problem of finding the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

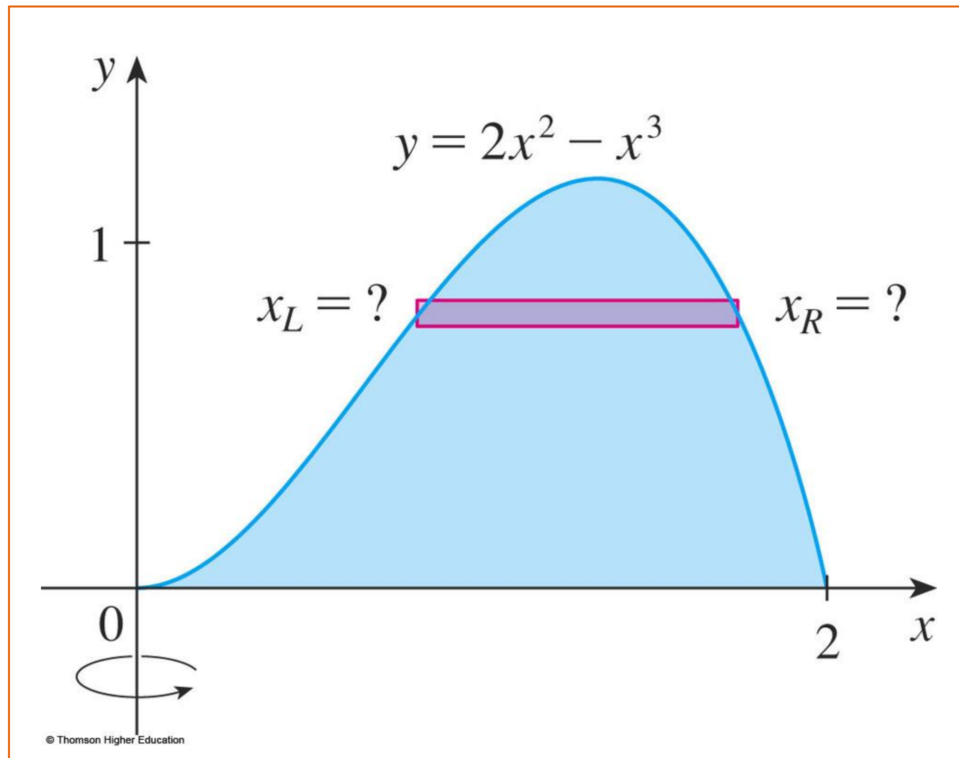


VOLUMES BY CYLINDRICAL SHELLS

If we slice perpendicular to the *y*-axis, we get a washer.

- However, to compute the inner radius and the outer radius of the washer, we would have to solve the cubic equation $y = 2x^2 - x^3$ for x in terms of y.

- That's not easy.



So, by the shell method,

the volume is:

$$V = \int_0^2 (2\pi x)\left(2x^2 - x^3\right)dx$$

$$= \int_0^2 (2\pi x)(2x^3 - x^4)dx$$

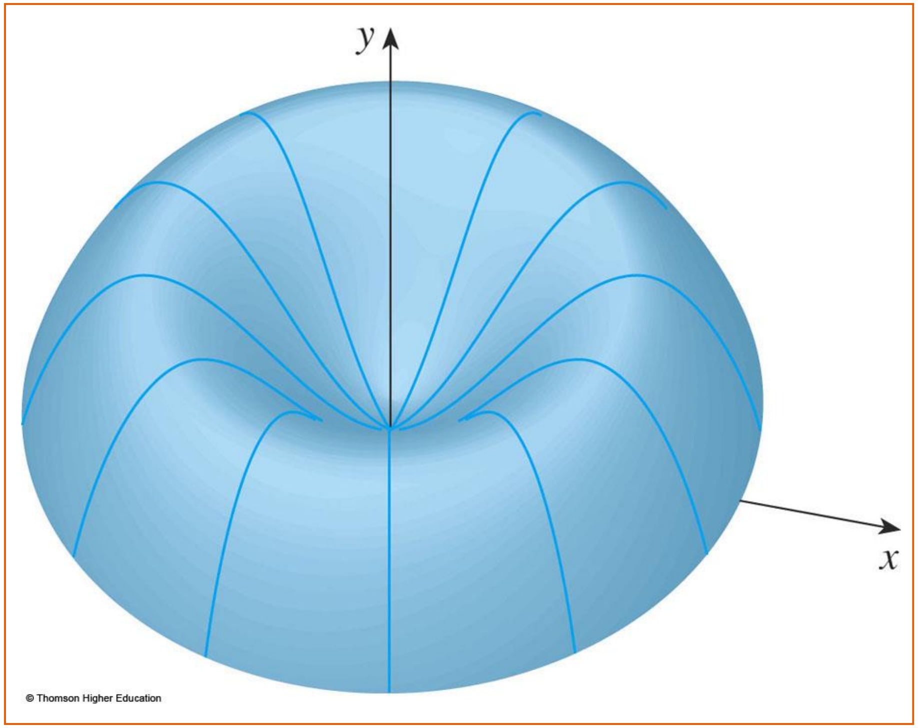
$$= 2\pi\left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5\right]_0^2$$

$$= 2\pi\left(8 - \tfrac{32}{5}\right) = \tfrac{16}{5}\pi$$

CYLINDRICAL SHELLS METHOD

Example 1

The figure shows a computer-generated picture of the solid whose volume we computed in the example.



When the strip is parallel to the axis of rotation, use the shell method.

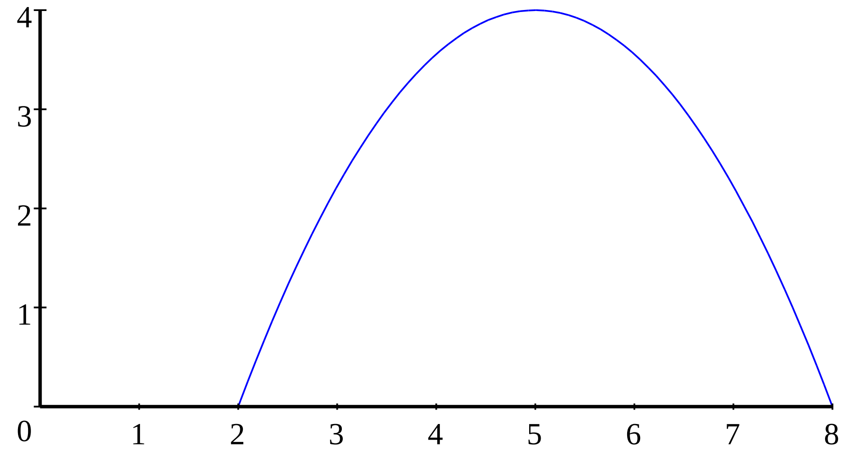
Şerit dönme eksenine paralel olduğunda, kabuk yöntemini kullanın.

When the strip is perpendicular to the axis of rotation, use the washer method.

Şerit dönme eksenine dik olduğunda, pul yöntemini kullanın.

Example 2

Find the volume generated when this shape is revolved about the y axis.

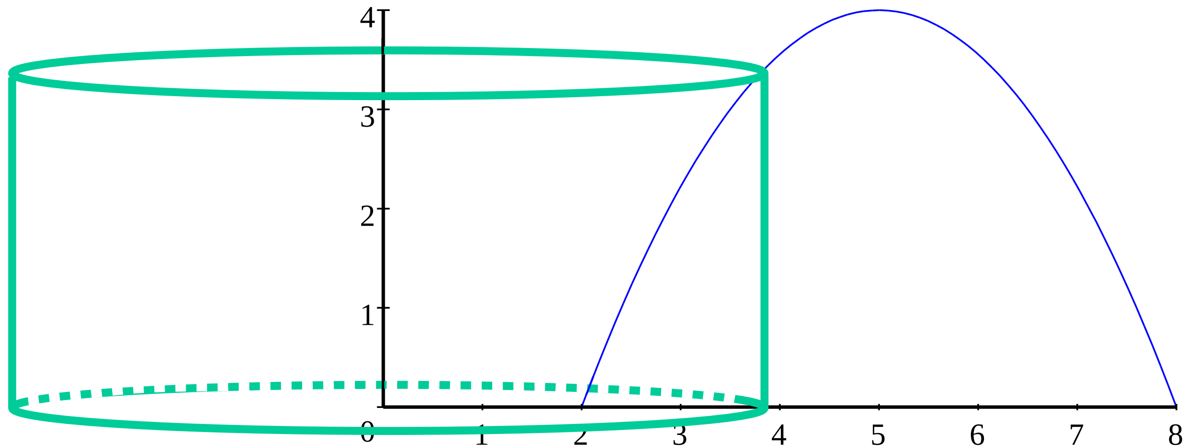


$$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$$

We can't solve for x, so we can't use a horizontal slice directly.

→

If we take a vertical slice and revolve it about the y-axis we get a cylinder.



$$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$$

Shell method:

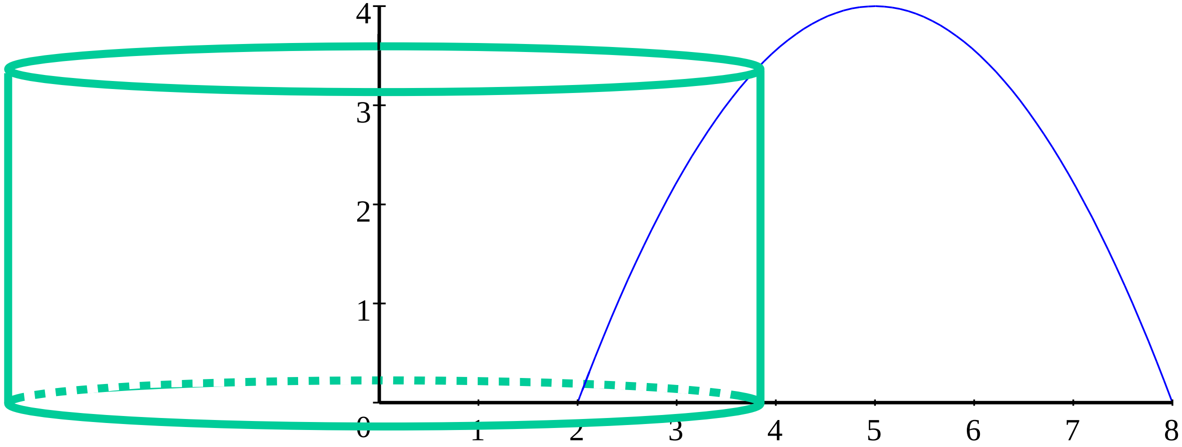
Lateral surface area of cylinder $\cdot$ thickness

$= \text{circumference} \cdot \text{height} \cdot \text{thickness}$

$= 2\pi r \cdot h \cdot dx$

Volume of thin cylinder $= 2\pi r \cdot h \cdot dx$

→



Volume of thin cylinder $= 2\pi r \cdot h \cdot dx$

$$y = -\frac{4}{9}\left(x^2 - 10x + 16\right)$$

$$\int_2^8 2\pi x\left[-\frac{4}{9}\left(x^2 - 10x + 16\right)\right] dx$$

$= 160\pi$

$\approx 502.655 \text{ cm}^3$

r — circumference

h

thickness

→

The tables below may be helpful:

Method	Axis of rotation	Integrate in
Disks and Washers	The x-axis The y-axis	x (use dx) y (use dy)
Cylindrical Shells	The x-axis The y-axis	y (use dy) x (use dx)

and

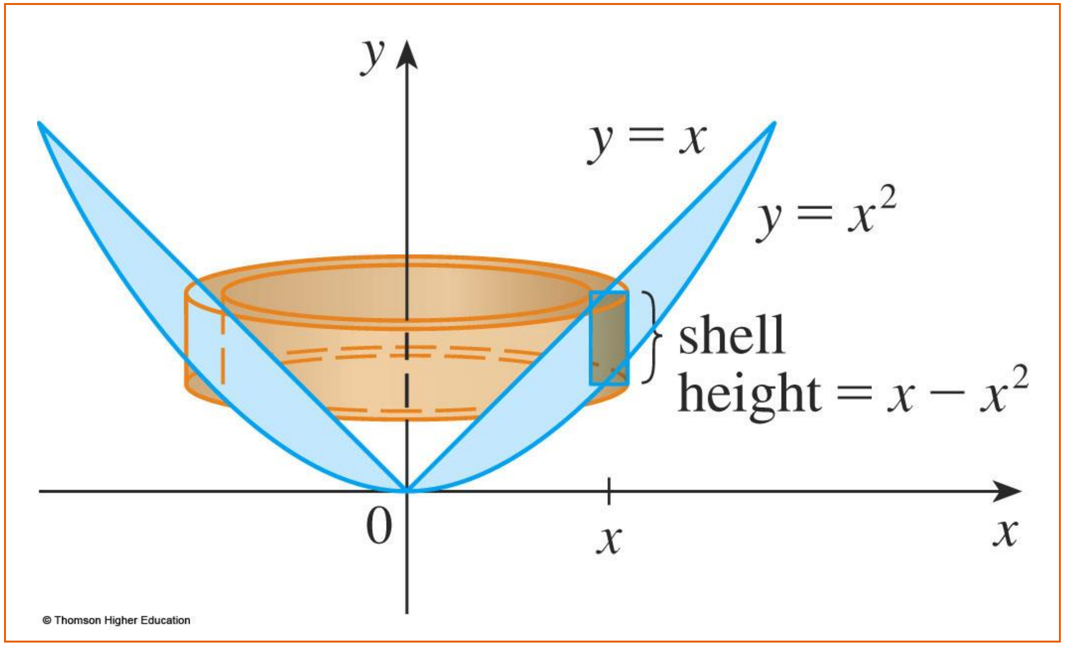
Method	Axis of rotation	formula
Disks and Washers	The x-axis	$V = \pi \int_a^b [f(x)]^2 \, dx$
	The y-axis	$V = \pi \int_a^b [f(y)]^2 \, dy$
Cylindrical Shells	The x-axis	$V = 2\pi \int_a^b y f(y) \, dy$
	The y-axis	$V = 2\pi \int_a^b x f(x) \, dx$

Example 3

Find the volume of the solid obtained by rotating about the *y*-axis the region between $y = x$ and $y = x^2$.

The region and a typical shell are shown here.

- We see that the shell has radius x, circumference $2\pi x$, and height $x - x^2$.



CYLINDRICAL SHELLS METHOD Example 3

- Thus, the volume of the solid is:

$$V = \int_0^1 (2\pi x)\left(x - x^2\right)dx$$

$$= 2\pi \int_0^1 \left(x^2 - x^3\right)dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

CYLINDRICAL SHELLS METHOD

As the following example shows, the shell method works just as well if we rotate about the *x*-axis.

- We simply have to draw a diagram to identify the radius and height of a shell.

CYLINDRICAL SHELLS METHOD

Example 4

Use cylindrical shells to find the volume of the solid obtained by rotating about the *x*-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

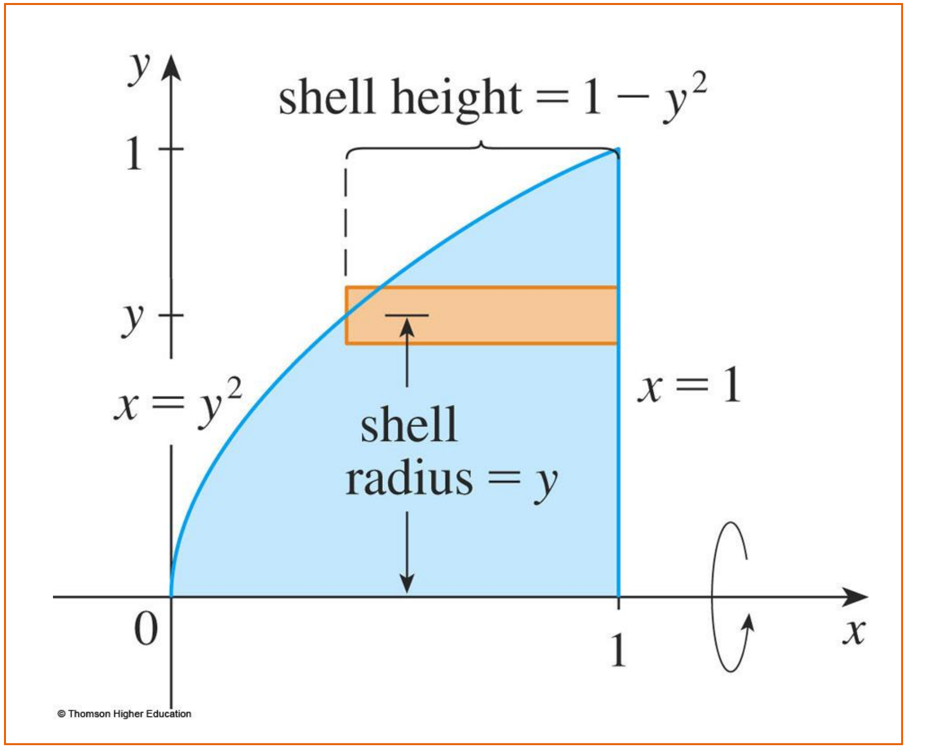
- **This problem was solved using disks before !!!**

CYLINDRICAL SHELLS METHOD   Example 3

To use shells, we relabel the curve

$y = \sqrt{x}$ as $x = y^2$.

- For rotation about the *x*-axis, we see that a typical shell has radius *y*, circumference $2\pi y$, and height $1 - y^2$.



CYLINDRICAL SHELLS METHOD Example 4

- So, the volume is:

$$V = \int_0^1 (2\pi y)\left(1 - y^2\right) dy$$

$$= 2\pi \int_0^1 (y - y^3) dy$$

$$= 2\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{\pi}{2}$$

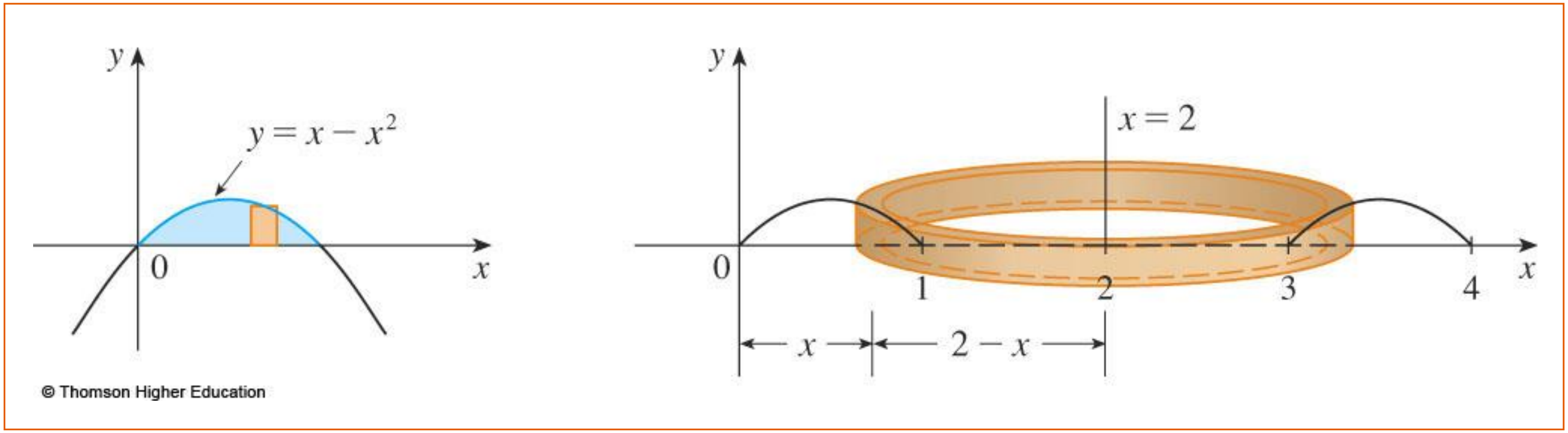
  – *In this problem, the disk method was simpler.*

CYLINDRICAL SHELLS METHOD Example 5

Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

CYLINDRICAL SHELLS METHOD Example 5

- The figures show the region and a cylindrical
- shell formed by rotation about the line $x = 2$,
- which has radius $2 - x$, circumference
- $2\pi(2 - x)$, and height $x - x^2$.



- So, the volume of the solid is:

$$V = \int_1^0 2\pi(2-x)\left(x-x^2\right)dx$$

$$= 2\pi\int_1^0 \left(x^3 - 3x^2 + 2x\right)dx$$

$$= 2\pi\left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 = \frac{\pi}{2}$$