A Review of Elementary Solution Methods

Here is a list of the kinds of equations I've discussed so far:

1. Separable equations.
2. Exact equations.
3. Homogeneous equations.
4. First-order linear equations.
5. Bernoulli and Riccati equations.
6. Equations requiring clever substitutions.
7. Linear constant coefficient homogeneous equations.

Linear constant coefficient homogeneous equations are straightforward, and I won't review them here. There are two things involved in solving the other types of equations:

1. You need to know *how* to apply each technique.
2. You need to know *which* technique to apply in a given problem.

Sometimes, it is simply a matter of trying one technique after another. However, this doesn't mean that you should use the first thing that works — there may be an easier way. Take the time to think about how each of the methods would work in a given problem.

Example. $(2x - 3y + 1)\, dx - (3x + 2y - 4)\, dy = 0$.

Write the equation as

$$(2x - 3y + 1)\, dx = (3x + 2y - 4)\, dy.$$

Evidently, there is no way to separate the x's and y's.

The equation is not homogeneous; however, it could be converted into a homogeneous equation by the substitutions $x = u + a$, $y = v + b$. After making the substitutions, you'd need to solve for a and b so as to make the constant terms vanish.

This method will work, though it is a little tedious.

Even when you have a method that will work, it is often wise to look at the problem a little longer to see if there is an easier way.

The equation does not seem to be first-order linear. On the other hand,

$$\frac{\partial M}{\partial y} = -3 \quad \text{and} \quad \frac{\partial N}{\partial x} = -3,$$

so the equation is exact. The method of exact equations is usually easier than the method of homogeneous equations, so I'll use exact equations rather than the substitution I noticed earlier.

I must find an f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate M with respect to x:

$$f = \int (2x - 3y + 1)\, dx = x^2 - 3xy + x + g(y).$$

Compute $\dfrac{\partial f}{\partial y}$ and set it equal to N:

$$-(3x + 2y - 4) = \frac{\partial f}{\partial y} = -3x + \frac{dg}{dy}.$$

Therefore,

$$\frac{dg}{dy} = -2y + 4, \quad g = -y^2 + 4y.$$

Therefore, $f = x^2 - 3xy + x - y^2 + 4y$. The solution is

$$x^2 - 3xy + x - y^2 + 4y = C. \quad \square$$

Example. $xy' = y + \sqrt{y^2 - x^2}$.

The equation is not separable. Solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x}.$$

It is not first-order linear in y.
Solve for $\dfrac{dx}{dy}$:

$$\frac{dx}{dy} = \frac{x}{y + \sqrt{y^2 - x^2}}.$$

It is not first-order linear in x.
Check for exactness. Write the equation as

$$(y + \sqrt{y^2 - x^2})\, dx - x\, dy = 0.$$

Then

$$\frac{\partial M}{\partial y} = 1 + \frac{y}{\sqrt{y^2 - x^2}} \quad \text{and} \quad \frac{\partial N}{\partial x} = -1.$$

It is not exact.
It better be homogeneous!

$$\frac{dy}{dx} = \frac{y + \sqrt{y^2 - x^2}}{x} = \frac{y}{x} + \sqrt{\left(\frac{y}{x}\right)^2 - 1}.$$

The right side is a function of $\dfrac{y}{x}$; the equation is homogeneous.
Let $y = vx$, so $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Then

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{v^2x^2 + x^2}}{x} = v + \sqrt{v^2 + 1}, \quad x\frac{dv}{dx} = \sqrt{v^2 + 1}.$$

Separate variables:

$$\int \frac{dv}{\sqrt{v^2 + 1}} = \int \frac{1}{x}\, dx, \quad \ln\left|\sqrt{v^2 + 1} + v\right| = \ln|x| + C.$$

I'll do the v-integral separately:

$$\int \frac{dv}{\sqrt{v^2 + 1}} = \int \frac{(\sec\theta)^2}{\sqrt{(\tan\theta)^2 + 1}}\, d\theta =$$

$$\int \frac{(\sec\theta)^2}{\sqrt{(\sec\theta)^2}}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\sqrt{v^2+1} + v| + C.$$

Put y back:

$$\ln\left|\sqrt{\left(\frac{y}{x}\right)^2 + 1} + \frac{y}{x}\right| = \ln|x| + C.$$

Exponentiate both sides and rename the constant:

$$\sqrt{\left(\frac{y}{x}\right)^2 + 1} + \frac{y}{x} = C_0 x. \quad \square$$

Example. $(3x^2y^3 - 2y)\,dx - x\,dy = 0$.

The equation is clearly not homogeneous or separable.

$$\frac{\partial M}{\partial y} = 9x^2y^2 - 2 \quad \text{but} \quad \frac{\partial N}{\partial x} = -1.$$

It is not exact.
Solve for $\frac{dy}{dx}$ and $\frac{dx}{dy}$:

$$\frac{dy}{dx} = 3xy^3 - \frac{2}{x}y \quad \text{and} \quad \frac{dx}{dy} = \frac{x}{3x^2y^3 - 2y}.$$

It is not first-order linear in x or y.
Rearrange the $\frac{dy}{dx}$ equation:

$$\frac{dy}{dx} + \frac{2}{x}y = 3xy^3.$$

It is a Bernoulli equation. Let $v = y^{1-3} = y^{-2}$. Then $\frac{dv}{dx} = -y^{-3}\frac{dy}{dx}$. Multiply the equation by $-2y^{-3}$:

$$-2y^{-3}\frac{dy}{dx} - \frac{4}{x}y^{-2} = -6x.$$

Substitute:

$$\frac{dv}{dx} - \frac{4}{x}v = -6x.$$

This is first order linear in v. The integrating factor is

$$I = \exp\int -\frac{4}{x}\,dx = x^{-4}.$$

Therefore,

$$vx^{-4} = \int -6x^{-3}\,dx = 3x^{-2} + C, \quad v = 3x^2 + Cx^4.$$

Put the y's back:

$$y^{-2} = 3x^2 + Cx^4, \quad y^2 = \frac{1}{3x^2 + Cx^4}. \quad \square$$

Example. $\frac{dy}{dx} = \tan y \cot x - \sec y \cos x$.

The equation is not first-order linear in either variable. It is not separable, nor is it homogeneous. It is not Bernoulli.

Is it exact? Rearrange it:

$$(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0.$$

Therefore,

$$\frac{\partial M}{\partial y} = -\cos y \cos x \quad \text{and} \quad \frac{\partial N}{\partial x} = \cos x \cos y.$$

It is not exact!

The idea here is to try to substitute to simplify the equation. The test of whether a substitution is the right one is whether it works! One rule of thumb is to look for common expressions — expressions that appear in several places. Another rule of thumb is to look for substitutions that eliminate one variable or another. In this vein, it is good to look for u-du combinations.

In the equation $(\sin x - \sin y)\cos x\, dx + \sin x \cos y\, dy = 0$ notice the "$\cos y\, dy$" at the end, the differential of $\sin y$. Try $u = \sin y$, so $du = \cos y\, dy$:

$$(\sin x - u)\cos x\, dx + \sin x\, du = 0, \quad \frac{du}{dx} - u\frac{\cos x}{\sin x} = -\cos x.$$

The equation is first-order linear in u.

The integrating factor is

$$I = \exp \int -\frac{\cos x}{\sin x}\, dx = \exp - \ln(\sin x) = \frac{1}{\sin x}.$$

Hence,

$$u\frac{1}{\sin x} = -\int \frac{\cos x}{\sin x}\, dx = -\ln|\sin x| + C.$$

Solve for u:

$$u = -\sin x \ln|\sin x| + C \sin x.$$

Put y back:

$$\sin y = -\sin x \ln|\sin x| + C \sin x. \quad \square$$

Example. A tank contains 20 gallons of pure water. Water containing 2 pounds of dissolved yogurt per gallon enters the tank at 4 gallons per minute. The well-stirred mixture drains out at 4 gallons per minute. How much yogurt is dissolved in the tank mixutre after 10 minutes? Find the limiting amount of yogurt in the tank as $t \to \infty$.

Let Y be the number of pounds of dissolved yogurt at time t.

$$\frac{dY}{dt} = \text{inflow} - \text{outflow} = \left(4\frac{\text{gal}}{\text{min}}\right)\left(2\frac{\text{lb}}{\text{gal}}\right) - \left(4\frac{\text{gal}}{\text{min}}\right)\left(\frac{Y\text{ lb}}{20\text{ gal}}\right).$$

Then

$$\frac{dY}{dt} + \frac{Y}{5} = 8.$$

You can do this by separation or by using an integrating factor. I will do the latter:

$$I = \exp \int \frac{1}{5}\, dt = \exp \frac{t}{5}.$$

Then

$$Y \exp \frac{t}{5} = \int \exp \frac{t}{5}\, dt = 40 \exp \frac{t}{5} + C.$$

The solution is

$$Y = 40 + C \exp \frac{-t}{5}.$$

Initially, there is no yogurt in the tank:

$$0 = Y(0) = 40 + C, \quad \text{so} \quad C = -40.$$

Therefore,

$$Y = 40 - 40 \exp \frac{-t}{5}.$$

When $t = 10$,

$$Y(10) = 40 - 40e^{-2} \approx 34.58659.$$

As $t \to \infty$, $\exp \frac{-t}{5} \to 0$, so $Y \to 40$. In the limit, the amount of dissolved yogurt approaches 40 pounds. This makes sense, since the tank is being flushed with water containing 2 pounds of yogurt per gallon, and the tank holds 20 gallons. □

