19/20 GÜZ MAT 201 FİNAL

CEVAP ANAHTARI

① $\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 3 & 2 & a & b \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -1 & a+3 & b \end{array}\right] \xrightarrow{\frac{-r_2}{3}\to r_2}$

$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & -1 & a+3 & b \end{array}\right] \xrightarrow{r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & a+2 & b-1 \end{array}\right]$

a) $a+2=0$ ve $b-1\neq 0 \Rightarrow a=-2$ ve $b\neq 1$ iken çözüm yok.

b) $a+2\neq 0 \Rightarrow a\neq -2$ iken tek çözüm.

c) $a+2=0$ ve $b-1=0 \Rightarrow a=-2$ ve $b=1$ iken sonsuz çözüm var.

② $A=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=4$ olmak üzere;

$A=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \overset{4c_3\to c_3}{=\!=} \frac{1}{4}\begin{vmatrix} a_1 & a_2 & 4a_3 \\ b_1 & b_2 & 4b_3 \\ c_1 & c_2 & 4c_3 \end{vmatrix} \overset{-2c_2+c_3\to c_3}{=\!=} \frac{1}{4}\begin{vmatrix} a_1 & a_2 & 4a_3-2a_2 \\ b_1 & b_2 & 4b_3-2b_2 \\ c_1 & c_2 & 4c_3-2c_2 \end{vmatrix}$

$\overset{\frac{r_3}{2}\to r_3}{=\!=} \frac{1}{4}\cdot 2\cdot\begin{vmatrix} a_1 & a_2 & 4a_3-2a_2 \\ b_1 & b_2 & 4b_3-2b_2 \\ \frac{c_1}{2} & \frac{c_2}{2} & 2c_3-c_2 \end{vmatrix} = \frac{1}{2}\cdot B$ (B, aradığımız determinant)

$A=\frac{1}{2}B \Rightarrow B=8.$

$(A=4)$

③ $S=\{v_1, v_2, v_3\} \subseteq \mathbb{R}^3$ için S 'nin sütun vektörlerini kullanarak kuracağımız matris A olsun.

Eğer $\det A \neq 0 \Rightarrow S$ kümesi lineer bağımsızdır.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & a & 1 \\ 1 & 0 & 2 \end{bmatrix}$$ olmak üzere 1. satıra göre açarak det hesaplayalım:

$$\det A = 1.(-1)^{1+1} . \begin{vmatrix} a & 1 \\ 0 & 2 \end{vmatrix} + (-1).(-1)^{1+2} . \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 2a - 1$$

O halde $\det A \neq 0$ olması için $a \neq \frac{1}{2}$ olmalıdır. Yani $a \neq \frac{1}{2}$ için S kümesi lineer bağımsızdır. $\mathbb{R}^3$ 'teki 3 elemanlı bir küme lineer bağımsız ise $\mathbb{R}^3$ 'ü gerer. O halde $a \neq \frac{1}{2}$ için S, $\mathbb{R}^3$ 'ün bir bazıdır.

④ $$A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$ için $$K_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ -1 & -1-\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

Determinantı 1. satıra göre açarak hesaplayalım:

$$= (2-\lambda).(-1)^{1+1} . \begin{vmatrix} -1-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} + (-1)^{1+2} . \begin{vmatrix} -1 & 1 \\ -1 & 2-\lambda \end{vmatrix} + (-1).(-1)^{1+3} . \begin{vmatrix} -1 & -1-\lambda \\ -1 & 1 \end{vmatrix}$$

$$= (2-\lambda)\left[(\lambda+1)(\lambda-2) - 1\right] - (\lambda - 2 + 1) - (-1 - 1 - \lambda)$$

$$= -\lambda^3 + 3\lambda^2 + \lambda - 3 = -(\lambda-3)(\lambda-1)(\lambda+1)$$

$\lambda_1 = 3$, $\lambda_2 = 1$, $\lambda_3 = -1$

3 farklı özdeğerimiz olduğundan

A, köşegenleştirilebilir.

$\underline{\lambda_1 = 3}$; $(A - \lambda I)x = 0$ sağlayan özvektörü bulalım: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$(A - 3I)x = 0$

$$\begin{bmatrix} -1 & 1 & -1 \\ -1 & -4 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[-r_1+r_3 \to r_3]{-r_1+r_2 \to r_2} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$(k \in \mathbb{R})$

$x_3 = k$

$x_2 = \frac{2k}{5}$

$x_1 = \frac{-3k}{5}$

$$\Rightarrow x = \left\{ k \begin{bmatrix} \frac{-3}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} ; k \in \mathbb{R} \right\}$$

$\underline{\lambda_2 = 1}$;

$$\begin{bmatrix} 1 & 1 & -1 \\ -1 & -2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{r_1+r_2 \to r_2} \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow[2r_2+r_3 \to r_3]{r_1+r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = k$

$x_2 = 0$

$x_1 = k$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \left\{ k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; k \in \mathbb{R} \right\}$$

$\underline{\lambda_3 = -1}$;

$$\begin{bmatrix} 3 & 1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow[3r_3+r_1 \to r_1]{-r_3+r_2 \to r_2} \begin{bmatrix} 0 & 4 & 8 \\ 0 & -1 & -2 \\ -1 & 1 & 3 \end{bmatrix} \xrightarrow{4r_2+r_1 \to r_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -2 \\ -1 & 1 & 3 \end{bmatrix} \longrightarrow$$

$$\xrightarrow[-r_2 \to r_2]{r_1 \leftrightarrow r_3} \begin{bmatrix} -1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[-r_2+r_1 \to r_1]{\frac{r_3}{16} \to r_3} \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = k$

$x_2 = -2k$

$x_1 = k$

$$\Rightarrow x = \left\{ k \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} ; k \in \mathbb{R} \right\}$$

Bu durumda $P^{-1}AP = D$ o.ş. P, D, P^{-1} ;

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{-3}{5} & 1 & 1 \\ \frac{2}{5} & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}.$$

P^{-1} matrisi herhangi bir yolla hesaplandığında aşağıdaki şekildedir.

$$P^{-1} = \begin{bmatrix} \frac{-5}{8} & 0 & \frac{5}{8} \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{-1}{8} & \frac{-1}{2} & \frac{1}{8} \end{bmatrix}$$ 'dir.

⑤
$$A=\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix} \qquad \det(A-\lambda I)=\begin{vmatrix}2-\lambda&1&-1\\1&2-\lambda&1\\-1&1&2-\lambda\end{vmatrix}$$

1. satıra göre açalım:

$$\det(A-\lambda I)=(2-\lambda)\begin{vmatrix}2-\lambda&1\\1&2-\lambda\end{vmatrix}-\begin{vmatrix}1&1\\-1&2-\lambda\end{vmatrix}-\begin{vmatrix}1&2-\lambda\\-1&1\end{vmatrix}=-\lambda(\lambda-3)^2$$

$\underline{\lambda_1=0}$;
$$\begin{bmatrix}2&1&-1\\1&2&1\\-1&1&2\end{bmatrix}\xrightarrow[r_3+r_2\to r_2]{2r_3+r_1\to r_1}\begin{bmatrix}0&3&3\\0&3&3\\-1&1&2\end{bmatrix}\xrightarrow[\frac{r_2}{3}\to r_2]{r_1\leftrightarrow r_3}\begin{bmatrix}-1&1&2\\0&1&1\\0&3&3\end{bmatrix}\longrightarrow$$

$$\xrightarrow[-r_2+r_1\to r_1]{-3r_2+r_3\to r_3}\begin{bmatrix}-1&0&1\\0&1&1\\0&0&0\end{bmatrix} \qquad \begin{matrix}x_3=k\\x_2=-k\\x_1=k\end{matrix} \quad \Rightarrow X=\left\{k\begin{bmatrix}1\\-1\\1\end{bmatrix};\ k\in\mathbb{R}\right\}\Rightarrow v_3=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$\underline{\lambda_{2,3}=3}$;
$$\begin{bmatrix}-1&1&-1\\1&-1&1\\-1&1&-1\end{bmatrix}\xrightarrow[r_2+r_1\to r_2]{r_3-r_1\to r_3}\begin{bmatrix}-1&1&-1\\0&0&0\\0&0&0\end{bmatrix} \qquad \begin{matrix}x_3=k\\x_2=m\\x_1=m-k\end{matrix} \quad \Rightarrow X=\left\{k\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{v_1}+m\underbrace{\begin{bmatrix}1\\1\\0\end{bmatrix}}_{v_2};\ k,m\in\mathbb{R}\right\}$$

Şimdi v_1 ve v_2'yi Gram-Smith'den ortogonal yapalım:

$u_1=v_1$

$$u_2=v_2-\frac{\langle v_2,u_1\rangle}{\langle u_1,u_1\rangle}\cdot u_1=\begin{bmatrix}1\\1\\0\end{bmatrix}-\frac{\left\langle\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}1\\1\\0\end{bmatrix}\right\rangle}{\left\langle\begin{bmatrix}-1\\0\\1\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\rangle}\cdot\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\1\\\frac{1}{2}\end{bmatrix}$$

Şimdi tüm vektörleri normalleştirelim:

$$w_1=\frac{u_1}{\|u_1\|}=\begin{bmatrix}\frac{-1}{\sqrt2}\\0\\\frac{1}{\sqrt2}\end{bmatrix},\quad w_2=\frac{u_2}{\|u_2\|}=\begin{bmatrix}\frac{1}{\sqrt6}\\\frac{2}{\sqrt6}\\\frac{1}{\sqrt6}\end{bmatrix},\quad w_3=\frac{v_3}{\|v_3\|}=\begin{bmatrix}\frac{1}{\sqrt3}\\\frac{-1}{\sqrt3}\\\frac{1}{\sqrt3}\end{bmatrix}$$

O halde;

$$D=\begin{bmatrix}3&0&0\\0&3&0\\0&0&0\end{bmatrix},\quad P=\begin{bmatrix}\frac{-1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\\0&\frac{2}{\sqrt6}&\frac{-1}{\sqrt3}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt6}&\frac{1}{\sqrt3}\end{bmatrix},\quad P^{-1}=P^T.$$