MAT 201 DOĞRUSAL CEBİR CEVAP ANAHTARI

1)

$$\begin{bmatrix} 3 & -4 & 5 \\ 6 & -2 & 4 \\ 7 & 1 & -11 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 16 \\ 50 \\ 56 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -4 & 5 & 16 \\ 6 & -2 & 4 & 50 \\ 7 & 1 & -11 & 56 \end{array}\right] \xrightarrow[-\frac{7}{3}r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 3 & -4 & 5 & 16 \\ 0 & 6 & -6 & 18 \\ 0 & \frac{31}{3} & -\frac{68}{3} & \frac{56}{3} \end{array}\right] \xrightarrow[\frac{r_3}{3}\to r_3]{\frac{r_2}{6}\to r_2} \left[\begin{array}{ccc|c} 3 & -4 & 5 & 16 \\ 0 & 1 & -1 & 3 \\ 0 & 31 & -68 & 56 \end{array}\right] \to$$

$$\xrightarrow[4r_2+r_1\to r_1]{-31r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 3 & 0 & 1 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -37 & -37 \end{array}\right] \xrightarrow{\frac{-r_3}{37}\to r_3} \left[\begin{array}{ccc|c} 3 & 0 & 1 & 28 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[r_3+r_2\to r_2]{-r_3+r_1\to r_1}$$

$$\to \left[\begin{array}{ccc|c} 3 & 0 & 0 & 27 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix}$$

2)

$$\begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & -1 \\ 2 & 2 & 2 \end{bmatrix} \xrightarrow[-2r_2+r_3\to r_3 \ (E_2)]{(E_1)\ -3r_2+r_1\to r_1} \begin{bmatrix} 0 & 2 & 3 \\ 1 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow[\frac{r_3}{4}\to r_3 \ (E_4)]{(E_3)\ r_1\leftrightarrow r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[-2r_3+r_2\to r_2 \ (E_6)]{(E_5)\ r_3+r_1\to r_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to$$

$$\xrightarrow[(E_7)]{-r_2+r_3\to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[(E_8)]{r_2\leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

($A \to I$ old. dan A matrisinin tersi vardır.)

$$A^{-1} = E_8 \cdot E_7 \cdots E_2 E_1$$

$$E_1 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \qquad E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}} \cdot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \cdot \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{4} \end{bmatrix}}$$

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} \\ -1 & \frac{3}{2} & \frac{3}{4} \\ 1 & -2 & \frac{-1}{2} \end{bmatrix}$$

(3)
$$\begin{vmatrix} x-a & b & -c & d \\ -a & x+b & -c & d \\ -a & b & x-c & d \\ -a & b & -c & x+d \end{vmatrix} \xlongequal[-r_1+r_4\to r_4]{\substack{-r_1+r_2\to r_2 \\ -r_1+r_3\to r_3}} \begin{vmatrix} x-a & b & -c & d \\ -x & x & 0 & 0 \\ -x & 0 & x & 0 \\ -x & 0 & 0 & x \end{vmatrix} =$$

$$\xlongequal{c_1+c_4\to c_1} \begin{vmatrix} x-a+d & b & -c & d \\ -x & x & 0 & 0 \\ -x & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} \xlongequal{c_1+c_3\to c_3} \begin{vmatrix} x-a+d-c & b & -c & d \\ -x & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix} =$$

$$\xlongequal{c_1+c_2\to c_2} \begin{vmatrix} x-a+d-c+b & b & -c & d \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{vmatrix}$$

$$= x^3(x-a+d-c+b)$$

(4) i) $A=\{1+t+2t^2,\ 1+3t-t^2,\ 1-t+t^2\}$ kümesinin P_2 'ye baz olabilmesi için $\text{Span}\,A = P_2$ olmalıdır. Kontrol edelim;

$$at^2+bt+c = c_1(1+t+2t^2)+c_2(1+3t-t^2)+c_3(1-t+t^2) \quad (\forall a,b,c \in \mathbb{R})$$

sağlayacak $c_1,c_2,c_3 \in \mathbb{R}$ var mıdır?

$$\begin{array}{ccc} t^2 & t & 1 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & a \\ -1 & 3 & 1 & b \\ 1 & -1 & 1 & c \end{array}\right] \xrightarrow[r_2+r_3\to r_3]{2r_2+r_1\to r_1} \left[\begin{array}{ccc|c} 0 & 7 & 3 & 2b+a \\ -1 & 3 & 1 & b \\ 0 & 2 & 2 & b+c \end{array}\right] \xrightarrow[\frac{r_3}{2}\to r_3]{r_1\leftrightarrow r_2} \left[\begin{array}{ccc|c} -1 & 3 & 1 & b \\ 0 & 7 & 3 & 2b+a \\ 0 & 1 & 1 & \frac{b+c}{2} \end{array}\right]$$

$$\xrightarrow[-3r_3+r_1\to r_1]{-7r_3+r_2\to r_2} \left[\begin{array}{ccc|c} -1 & 0 & -2 & \frac{-b-3c}{2} \\ 0 & 0 & -4 & \frac{2a-3b-7c}{2} \\ 0 & 1 & 1 & \frac{b+c}{2} \end{array}\right] \xrightarrow[-\frac{r_2}{4}\to r_2]{-r_1\to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{b+3c}{2} \\ 0 & 0 & 1 & \frac{3b+7c-2a}{8} \\ 0 & 1 & 1 & \frac{b+c}{2} \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3}$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & \frac{b+3c}{2} \\ 0 & 1 & 1 & \frac{b+c}{2} \\ 0 & 0 & 1 & \frac{3b+7c-2a}{8} \end{array}\right] \xrightarrow[-r_3+r_2\to r_2]{-2r_3+r_1\to r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2a-b-c}{4} \\ 0 & 1 & 0 & \frac{2a+b-3c}{8} \\ 0 & 0 & 1 & \frac{3b+7c-2a}{8} \end{array}\right]$$

o halde;

$$\left.\begin{aligned} c_1 &= \frac{2a-b-c}{4} \\ c_2 &= \frac{2a+b-3c}{8} \\ c_3 &= \frac{3b+7c-2a}{8} \end{aligned}\right\} \quad a,b,c\in\mathbb{R} \text{ old. } c_1,c_2,c_3 \in \mathbb{R} \text{ 'dir.}$$

$$\Rightarrow \text{Span}\,A = P_2 .$$

A kümesinin P_2 'ye baz olabilmesi için A'nın vektörlerinin lin. bağımsız olması gerekir. Yani yukarıdaki denklemde $a=b=c=0$ iken $c_1=c_2=c_3=0$ olması gerekir.

Gerçekten $a=b=c=0$ iken

$c_1=c_2=c_3=0$ olduğundan

A kümesi P_2 için bazdır.

④ ii) $V = \mathbb{R}^+$. V üzerinde top. skalerle çarpma işlemleri $u, v \in V, d\, c \in \mathbb{R}$

$$u \oplus v = uv \quad , \quad c \odot v = v^{c-1}$$

için $(V, \oplus, \odot)$ vektör uzay olma koşullarını inceleyelim :

1) $u \oplus v = uv \overset{\mathbb{R}\text{'de çarpma değ.}}{=} v.u = v \oplus u$

2) $u \oplus (v \oplus w) = u(vw) = (uv)w = (u \oplus v) \oplus w$

3) $u \oplus 0_V = u.0_V = u \Rightarrow 0_V = 1 \in \mathbb{R}^+$

4) $u \oplus u^{-1} = 0_V \Rightarrow u^{-1} = \frac{1}{u} \in \mathbb{R}^+$

5) $c \odot (u \oplus v) = c \odot (uv) = (uv)^{c-1} = u^{c-1} . v^{c-1} = (c \odot u) \oplus (c \odot v)$

6) $(c+d) \odot u = u^{c+d-1} \neq u^{c-1} . u^{d-1} = (c \odot u) \oplus (d \odot u)$

O halde $(V, \oplus, \odot)$ vektör uzayı değildir.

5) i) $A = \begin{bmatrix} 5 & 1 & 0 \\ -8 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}$, $K_A(\lambda) = \begin{vmatrix} 5-\lambda & 1 & 0 \\ -8 & -\lambda & 1 \\ 4 & 0 & -\lambda \end{vmatrix}$

$$K_A(\lambda) \overset{\text{1. satıra göre}}{=} (5-\lambda)(-1)^{1+1} \cdot \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} + 1(-1)^{1+2} \cdot \begin{vmatrix} -8 & 1 \\ 4 & -\lambda \end{vmatrix} = -(\lambda-1)(\lambda-2)^2$$

Özvektörler: $(A-\lambda I)x=0$

$\lambda=2$: $\begin{bmatrix} 3 & 1 & 0 \\ -8 & -2 & 1 \\ 4 & 0 & -2 \end{bmatrix} \xrightarrow{2r_3+r_2 \to r_2} \begin{bmatrix} 3 & 1 & 0 \\ 0 & -2 & -3 \\ 4 & 0 & -2 \end{bmatrix} \xrightarrow[\frac{r_2}{-2} \to r_2]{\frac{-4r_1}{3}+r_3 \to r_3} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & \frac{-4}{3} & -2 \end{bmatrix}$

$\xrightarrow{\frac{4}{3}r_2+r_3 \to r_3} \begin{bmatrix} 3 & 1 & 0 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x = \left\{ m \begin{bmatrix} \frac{1}{2} \\ \frac{-3}{2} \\ 1 \end{bmatrix} ; m \in \mathbb{R} \right\}$

$\lambda=2$ çakışık özdeğeri için tek özvektör old.dan A, köşegenleştirilemez.

ii) $A = A^T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $K_A(\lambda) = \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 1-\lambda & 2 \\ -1 & 2 & 1-\lambda \end{vmatrix}$

1. satıra göre açarsak

$$K_A(\lambda) = (2-\lambda)(-1)^{1+1} \cdot \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} + (-1)(-1)^{1+2} \cdot \begin{vmatrix} -1 & 2 \\ -1 & 1-\lambda \end{vmatrix} + (-1)(-1)^{1+3} \cdot \begin{vmatrix} -1 & 1-\lambda \\ -1 & 2 \end{vmatrix}$$

$= -(\lambda-1)(\lambda-4)(\lambda+1)$ özdeğerler; $\lambda_1=1$, $\lambda_2=4$, $\lambda_3=-1$

Özvektörler: $(A-\lambda I)X=0$

$\lambda_1=1$: $\begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix} \xrightarrow[r_1+r_3 \to r_3]{r_1+r_2 \to r_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[r_1-r_2 \to r_1]{r_3+r_2 \to r_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow X_1 = \left\{ m \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} ; m \in \mathbb{R} \right\}$

$\lambda_2=4$: $\begin{bmatrix} -2 & -1 & -1 \\ -1 & -3 & 2 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow[-r_3+r_2 \to r_2]{-2r_3+r_1 \to r_1} \begin{bmatrix} 0 & -5 & 5 \\ 0 & -5 & 5 \\ -1 & 2 & -3 \end{bmatrix} \xrightarrow{-r_2+r_1 \to r_1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -5 & 5 \\ -1 & 2 & -3 \end{bmatrix} \Rightarrow X_2 = \left\{ m \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} ; m \in \mathbb{R} \right\}$

$\underline{\lambda_3=-1}$: $\begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & 2 \\ -1 & 2 & 2 \end{bmatrix} \xrightarrow[-r_2+r_3\to r_3]{3r_2+r_1\to r_1} \begin{bmatrix} 0 & 5 & 5 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_3 = \left\{ m\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} ; m \in \mathbb{R} \right\}$

* özdeğerler çakışık olmadığından bulduğumuz x_1, x_2, x_3 özvektörler ortogonaldir.

Şimdi bu vektörleri ortonormal yapalım:

$$w_1 = \frac{x_1}{\|x_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \quad w_2 = \frac{x_2}{\|x_2\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad w_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad P^T = P^{-1}.$$